SCATTERING FOR THE FOCUSING L^2 -SUPERCRITICAL AND $\dot{H}^2$-SUBCRITICAL BIHARMONIC NLS EQUATIONS

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ABSTRACT. We consider the focusing $\dot{H}^{s_c}$-critical biharmonic Schrödinger equation, and prove a global wellposedness and scattering result for the radial data $u_0 \in H^2(\mathbb{R}^N)$ satisfying $M(u_0)^{\frac{2-s_c}{s_c}} E(u_0) < M(Q)^{\frac{2-s_c}{s_c}} E(Q)$ and $\|u_0\|_2^{\frac{2-s_c}{s_c}} \|\Delta u_0\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}} \|\Delta Q\|_2$, where $s_c \in (0, 2)$ and Q is the ground state of $\Delta^2 Q + (2 - s_c)Q - |Q|^{p-1}Q = 0$.

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1. Introduction

The biharmonic Schrödinger equations, which are also called the fourth-order Schrödinger equations,

$$iu_t + \Delta^2 u - \varepsilon \Delta u + f(|u|^2)u = 0 \tag{1.1}$$

with $\varepsilon = \pm 1$ or $\varepsilon = 0$ were introduced by Karpman [14] and Karpman and Shagalov [15] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Sharp dispersive estimates for the biharmonic Schrödinger operator have recently been obtained in [1], while specific nonlinear biharmonic Schrödinger equations as (1.1) were discussed in [8, 10, 7, 29]. Related equations also appeared in [5, 11, 30]. For a pure power-type nonlinearity, i.e., $f(|u|^2)u = \mu|u|^{p-1}u$, the equation (1.1) is subcritical for $N \le 4$ or for $p < 1 + \frac{8}{N-4}$ $(N > 4)$. When $N \ge 5$, criticality in the energy space $H^2(\mathbb{R}^N)$ appears with the power $p = 1 + \frac{8}{N-4}$. Fibich, Ilan and Papanicolaou in [7] describe various properties of the equation in the subcritical regime, with part of their analysis relying on very interesting numerical developments. Segata in [29] proved scattering in $\mathbb{R}^1$ for the cubic nonlinearity; while in higher dimensions, the scattering results were obtained in [22, 26]. Global well-posedness and scattering for the energy critical case was considered in [27, 25, 24, 23], while in [28], the authors proved the same results for the mass-critical fourth-order Schrödinger equation in high dimensions. As discussed in [27], the scattering results for the subcritical defocusing case, i.e., for $f(|u|^2)u = |u|^{p-1}u$ with $1 + \frac{8}{N} < p < 1 + \frac{8}{N-4}$, could be obtained following the strategy in Lin and Strauss [20], see also [3]. However, to the authors' knowledge, there have not been any scattering results for the focusing case $(f(|u|^2)u = -|u|^{p-1}u)$ in the subcritical regime.

In this paper, we consider the focusing L^2 -supercritical and $\dot{H}^2$-subcritical biharmonic nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta^2 u - |u|^{p-1}u = 0, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ \qquad\qquad u(x,0) = u_0(x) \in H^2(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $u(x,t)$ is a complex-valued function in $\mathbb{R}^N \times \mathbb{R}$ with the space dimension N satisfying $1+\frac{8}{N} < p < 1+\frac{8}{N-4}$ (when $N \leq 4$, $1+\frac{4}{N} < p < \infty$). Equation (1.2) admits two important conservation laws in energy space $H^2(\mathbb{R}^N)$:

$$Mass: \quad M(u)(t) \equiv \int |u(x,t)|^2 dx = M(u_0);$$

$$Energy: \quad E(u)(t) \equiv \frac{1}{2}\int |\Delta u(x,t)|^2 dx - \frac{1}{p+1}\int |u(x,t)|^{p+1} dx = E(u_0).$$

Moreover, it is easy to check that equation (1.2) is invariant under the scaling $u(x,t) \to \lambda^{\frac{4}{p-1}} u(\lambda x, \lambda^2 t)$ which also leaves the norm of the homogeneous Sobolev space $\dot{H}^{s_c}(\mathbb{R}^N)$ invariant, where $s_c \in (0,2)$ is defined by $s_c = \frac{N}{2} - \frac{4}{p-1}$. That is why we also call this equation $\dot{H}^{s_c}$-critical. Other scaling invariant quantities are $\|\Delta u\|_{L^2(\mathbb{R}^N)} \|u\|_{L^2(\mathbb{R}^N)}^{\frac{2-s_c}{s_c}}$ and $E(u)M(u)^{\frac{2-s_c}{s_c}}$.

Equation (1.2) possesses a focusing nonlinearity ($f(|u|^2)u = -|u|^{p-1}u$), so one cannot hope to a similar global result as in [27]. Indeed, the existence of a nontrivial solution of the elliptic equation

$$\Delta^2 Q + (2-s_c)Q - |Q|^{p-1}Q = 0, \tag{1.3}$$

which we refer to as ground state $Q \in H^2(\mathbb{R}^N)$, can be obtained by similar method to that used in [2]. We then conclude that solitary waves $u(x,t) = e^{i(2-s_c)t}Q(x)$ do not scatter. One can refer to [7] for some similar results.

Our aim in this paper is to obtain the following result of scattering for the solutions of (1.2) with radial data. This scattering result would complement the very recent analysis of Boulenger and Lenzmann [2].

Theorem 1.1. *Let $u_0 \in H^2$ be radial and let u be the corresponding solution to* (1.2) *with maximal forward time interval of existence $I \subset \mathbb{R}$. Suppose $M(u_0)^{\frac{2-s_c}{s_c}} E(u_0) < M(Q)^{\frac{2-s_c}{s_c}} E(Q)$, where Q is the solution of* (1.3). *If*

$$\|u_0\|_{L^2(\mathbb{R}^N)}^{\frac{2-s_c}{s_c}} \|\Delta u_0\|_{L^2(\mathbb{R}^N)} < \|Q\|_{L^2(\mathbb{R}^N)}^{\frac{2-s_c}{s_c}} \|\Delta Q\|_{L^2(\mathbb{R}^N)},$$

then $I = (-\infty, +\infty)$, and u scatters in $H^2(\mathbb{R}^N)$. That is there exists $\phi_\pm \in H^2(\mathbb{R}^N)$ such that $\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta^2}\phi_\pm\|_{H^2(\mathbb{R}^N)} = 0$.

Our paper is organized as follows. We fix notations in the end of section 1. In section 2, We recall the local theory for (1.2) established by [27]. After that, we introduce the inhomogeneous Strichartz's estimates, upon which we sketch the proof of the small data

scattering and the perturbation theory. The variational structure of the ground state of an elliptic problem is given in section 3. In section 4, we prove a dichotomy proposition of global well-posedness versus blowing up, which yields the comparability of the total energy and the kinetic energy. The concentration compactness principle is used in section 5 to give a critical element, which yields a contradiction through a virial-type estimate in section 6, concluding the proof of Theorem 1.1.

Notations: In what follows, we denote by c a generic constant that is allowed to depend on N and p. The exact value of that constant may change from one to another. We write C_a when there is more dependence. We let $L^r = L^r(\mathbb{R}^N)$ be the usual Lebesgue spaces with the norm defined by $\|\cdot\|_r$, and $L^q(I, L^r)$ be the spaces of measurable functions from an interval $I \subset \mathbb{R}$ to L^q whose $L^q(I, L^r)$-norm is finite, where $L^q(I, L^r) = \left(\int_I \|u(t)\|_r^q dt\right)^{\frac{1}{q}}$. Moreover, we define the Fourier transform on $\mathbb{R}^N$ by $\hat{f}(\xi) = (2\pi)^{-N/2} \int e^{-ix\xi} f(x)dx$. For $s \in \mathbb{R}$, the pseudo-differential operator $(-\Delta)^s$ (or denoted by $|\nabla|^{2s}$) is defined by $\widehat{(-\Delta)^s f}(\xi) \equiv |\xi|^{2s}\hat{f}(\xi)$, which in turn defines the homogeneous Sobolev space $\dot{H}^s = \dot{H}^s(\mathbb{R}^N) \equiv \left\{ f \in \mathcal{S}'(\mathbb{R}^N) : \int |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi < \infty \right\}$ with its norm defined by $\|f\|_{\dot{H}^s} = \|(-\Delta)^s f\|_2$, where $\mathcal{S}'(\mathbb{R}^N)$ denotes the space of tempered distributions.

2. Local theory and Strichartz estimates

We start in this section by recalling the Strichartz estimates established by Pausader [27]. We say a pair (q, r) is Schrödinger admissible, for short S-admissible, if $2 \le q, r \le \infty$, $(q, r, N) \ne (2, \infty, 2)$, and $\frac{2}{q} + \frac{N}{r} = \frac{N}{2}$. Also we use the terminology that a pair (q, r) is biharmonic admissible, for short B-admissible, if $2 \le q, r \le \infty$, $(q, r, N) \ne (2, \infty, 4)$, and $\frac{4}{q} + \frac{N}{r} = \frac{N}{2}$.

The Strichartz estimates are stated as follows. Let $u \in C(I, H^{-4}(\mathbb{R}^N))$ be a solution of

$$u(t) = e^{it(\Delta^2+\varepsilon\Delta)}u_0 + i\int_0^t e^{i(t-s)(\Delta^2+\varepsilon\Delta)}h(s)ds \tag{2.1}$$

with $\varepsilon \in \{-1, 0, 1\}$ on an interval $I = [0, T]$. If $\varepsilon = 1$, suppose also $|I| \le 1$. For any B-admissible pairs (q, r) and $(\bar{q}, \bar{r})$,

$$\|u\|_{L^q(I,L^r)} \le c(\|u_0\|_{L^2} + \|h\|_{L^{\bar{q}'}(I,L^{\bar{r}'})}), \tag{2.2}$$

where $\bar{q}'$ and $\bar{r}'$ are the conjugate exponents of $\bar{q}$ and $\bar{r}$, i.e., $\frac{1}{\bar{q}} + \frac{1}{\bar{q}'} = \frac{1}{\bar{r}} + \frac{1}{\bar{r}'} = 1$. Besides, for any S-admissible pairs (q, r) and (a, b), and any $s \ge 0$,

$$\||\nabla|^s u\|_{L^q(I,L^r)} \le c(\||\nabla|^{s-\frac{2}{q}}u_0\|_{L^2} + \||\nabla|^{s-\frac{2}{q}-\frac{2}{a}}h\|_{L^{a'}(I,L^{b'})}). \tag{2.3}$$

From Sobolev embedding, estimates (2.3) implies (2.2). We define two norms for convenience to study the $\dot{H}^{s_c}$-critical equation (1.2) by

$$\|u\|_{Z(I)} = \|u\|_{L^{\frac{(N+4)(p-1)}{4}}(I,L^{\frac{(N+4)(p-1)}{4}})}, \qquad \|u\|_{Z'(I)} = \|u\|_{L^{\frac{(N+4)(p-1)}{4p}}(I,L^{\frac{(N+4)(p-1)}{4p}})}. \tag{2.4}$$

Thus a direct consequence of (2.3) and the Sobolev's inequality is that, if $u \in C(I, H^{-4}(\mathbb{R}^N))$ be a solution of (2.1) with $u_0 \in \dot{H}^2$ and $\nabla h \in L^2(I, L^{\frac{2N}{N+2}})$, then $u \in C(I, \dot{H}^2(\mathbb{R}^N))$ and for any B-admissible pairs (q, r),

$$\|\Delta u\|_{L^q(I,L^r)} \leq c(\|\Delta u_0\|_{L^2} + \|\nabla h\|_{L^2(I,L^{\frac{2N}{N+2}})}). \tag{2.5}$$

A key feature of (2.5) is that the second derivative of u is estimated using only one derivative of the forcing term h. Just as in [27], the Strichartz estimates yield the following local well-posedness result.

Proposition 2.1. *Given any initial data $u_0 \in H^2$, any $p \in (1, \frac{N+4}{N-4})$ when $N \geq 5$, and any $p > 1$ when $N \leq 4$, there exists $T > 0$ and a unique solution $u \in C([0,T], H^2)$ of (1.2) with initial data u_0. The solution has conserved mass and energy. Besides, if T^+ is the maximal time of existence of u, then $\lim_{t \to T^+} \|u(t)\|_{H^2} = +\infty$ when $T^+ < \infty$. And the solution map $u_0 \mapsto u$ is continuous in the sense that for any $T \in (0, T^+)$, if $u_0^k \in H^2$ is a sequence converging in H^2 to u_0, and if u^k denotes the solution of (1.2) with initial data u_0^k, then u^k is defined on $[0,T]$ for sufficiently large k and $u^k \to u$ in $C([0,T], H^2)$.*

The author in [31] (Theorem 1.4 there) studied the inhomogeneous Strichartz estimates for dispersive operators in the abstract setting, which indeed yield the counterpart ones for the fourth-order Schrödinger operators by the same method used in the proof of Corollary 7.1 in [31]. More precisely, we say that a pair (q, r) is $\dot{H}^s$-biharmomic admissible and denote it by $(q, r) \in \Lambda_s$ if $0 \leq s < 2$ and

$$\frac{4}{q} + \frac{N}{r} = \frac{N}{2} - s, \quad \frac{2N}{N-2s} \leq r < \frac{2N}{N-4}.$$

Correspondingly, we call the pair (q', r') dual $\dot{H}^s$-biharmomic admissible, denoted by $(q', r') \in \Lambda'_s$, if $(q, r) \in \Lambda_{-s}$ and (q', r') is the conjugate exponent pair of (q, r). In particular, $(q, r) \in \Lambda_0$ is just a B-admissible pair, which is always denoted by $(q, r) \in \Lambda_B$. Combining the results obtained by [27] and [31], we can infer the following inhomogeneous Strichartz estimates on $I = [0, T]$:

$$\left\| \int_0^t e^{i(t-t^1)\Delta^2} f(\cdot, s) ds \right\|_{L^q(I;L^r)} \leq c\|f\|_{L^{\tilde{q}'}(I;L^{\tilde{r}'})}, \quad \forall \ (q,r) \in \Lambda_s, \ \forall \ (\tilde{q}, \tilde{r}) \in \Lambda_{-s}. \tag{2.6}$$

We also refer to [6, 16, 17, 32] for more precise discussion on the inhomogeneous Strichartz estimates.

Note that in the definition of $\|\cdot\|_{Z(I)}$ and $\|\cdot\|_{Z'(I)}$, the pair $(\frac{(N+4)(p-1)}{4}, \frac{(N+4)(p-1)}{4}) \in \Lambda_{s_c}$ and $(\frac{(N+4)(p-1)}{4p}, \frac{(N+4)(p-1)}{4p}) \in \Lambda'_{s_c}$ with $s_c = \frac{2}{N} - \frac{4}{p-1}$ defined in the introduction. Since for any $(q, r) \in \Lambda_{s_c}$, we can check that $(\frac{q}{p}, \frac{r}{p}) \in \Lambda'_{s_c}$, then the inhomogeneous Strichartz estimates (2.6) combined with the Hölder inequality give that

$$\left\| \int_0^t e^{i(t-s)\Delta^2} |u|^{p-1} u(s) ds \right\|_{L^q(I;L^r)} \leq c\| |u|^{p-1} u\|_{L^{\frac{q}{p}}(I;L^{\frac{r}{p}})} \leq c \|u\|^p_{L^q(I;L^r)}. \tag{2.7}$$

As a consequence of the Strichartz estimates introduced above, we can obtain the following proposition.

Proposition 2.2. *Assume $u_0 \in H^2$, $t_0 \in I$ an interval of $\mathbb{R}$. Then there exists $\delta_{sd} > 0$ such that if $\|e^{it\Delta^2}u_0\|_{Z(I)} \le \delta_{sd}$, then there exists a unique solution $u \in C(I, H^2)$ of (1.2) with initial data u_0. This solution has conserved mass and energy, and satisfies*

$$\|u\|_{Z(I)} \le 2\delta_{sd}, \quad \|u\|_{L^\infty(I,H^2)} \le c\|u_0\|_{H^2}. \tag{2.8}$$

Proof. For $\delta = \delta_{sd}$ and $M = c\|u_0\|_{H^2}$, we define a map as

$$\Phi(u) = e^{i(t-t_0)\Delta^2}u_0 + i\int_{t_0}^{t} e^{i(t-s)\Delta^2}|u|^{p-1}u(s)ds,$$

and a set as

$$M_{M,\delta} = \{v \in C(I, H^2) : \quad \|v\|_{Z(I)} \le 2\delta, \quad \|v\|_{L^{2(p-1)}(I,L^{\frac{N(p-1)}{2}})} \le 2\delta, \quad \|\Delta v\|_{L^\infty(I,L^2)} \le 2M\}$$

equipped with the $Z(I)$ norm. Then from the Strichartz estimates (2.5) and (2.7), using the Sobolev embedding and the Hölder inequalities, we have for any $u \in M_{M,\delta}$,

$$\|\Phi(u)\|_{Z(I)} \le \delta + c\|u\|^p_{Z(I)}, \qquad \|\Phi(u)\|_{L^{2(p-1)}(I,L^{\frac{N(p-1)}{2}})} \le \delta + c\|u\|^p_{Z(I)}$$

and

$$\begin{aligned}\|\Delta\Phi(u)\|_{L^\infty(I,L^2)} &\le c\|\Delta u_0\|_2 + c\|u\|^{p-1}_{L^{2(p-1)}(I,L^{\frac{N(p-1)}{2}})}\|\nabla u\|_{L^\infty(I,L^{\frac{2N}{N-2}})}\\ &\le c\|\Delta u_0\|_2 + c\|u\|^{p-1}_{L^{2(p-1)}(I,L^{\frac{N(p-1)}{2}})}\|\Delta u\|_{L^\infty(I,L^2)}.\end{aligned}$$

Moreover, for any $u, v \in M_{M,\delta}$,

$$\|\Phi(u) - \Phi(v)\|_{Z(I)} \le c(\|u\|^{p-1}_{Z(I)} + \|v\|^{p-1}_{Z(I)})\|u - v\|_{Z(I)}.$$

From a standard argument, we can obtain that if δ is sufficiently small, the map $u \mapsto \Phi(u)$ is a contraction map on $M_{M,\delta}$. Thus, the contraction mapping theorem gives a unique solution u in $M_{M,\delta}$ satisfying (2.8). $\square$

From the small data theory (Proposition 2.2) and using a similar argument as in [27], we can obtain the following result of scattering, the proof of which is standard and we omit here.

Proposition 2.3. *Let $u(t) \in C(\mathbb{R}, H^2)$ be a solution of (1.2). If $\|u\|_{Z(\mathbb{R})} < \infty$, then $u(t)$ scatters in H^2. That is , there exists $\phi^\pm \in H^2$ such that $\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}\phi^\pm\|_{H^2} = 0$.*

Now we show a useful perturbation lemma as follows.

Lemma 2.4. *For any given A, there exist $\epsilon_0 = \epsilon_0(A, N, p)$ and $c = c(A)$ such that for any $\epsilon \le \epsilon_0$, any interval $I = (T_1, T_2) \subset \mathbb{R}$ and any $\tilde{u} = \tilde{u}(x, t) \in H^2$ satisfying*

$$i\tilde{u}_t + \Delta^2\tilde{u} - |\tilde{u}|^{p-1}\tilde{u} = e,$$

if for some $(q, r) \in \Lambda_{-s_c}$,

$$\|\tilde{u}\|_{Z(I)} \le A, \quad \|e\|_{L^{q'}(I;L^{r'})} \le \epsilon$$

and

$$\|e^{i(t-t_0)\Delta^2}(u(t_0)-\tilde{u}(t_0)\|_{Z(I)} \le \epsilon,$$

then the solution $u \in C(I;H^2)$ *of* (1.2) *satisfying*

$$\|u-\tilde{u}\|_{Z(I)} \le c(A)\epsilon.$$

Proof. Let w be defined by $u = \tilde{u} + w$. Then w solves the equation

$$i\partial_t w + \Delta^2 w - |w+\tilde{u}|^{p-1}(w+\tilde{u}) + |\tilde{u}|^{p-1}\tilde{u} + e = 0. \tag{2.9}$$

For any $t_0 \in I$, $I = (T_1, t_0] \cup [t_0, T_2)$. We need only consider on $I_+ = [t_0, T_2)$, since the case on $I_- = (T_1, t_0]$ can be considered similarly. Since $\|\tilde{u}\|_{Z(I)} \le A$, we can partition $[t_0, T_2)$ into $N = N(A)$ intervals $I_j = [t_j, t_{j+1}]$ such that for each j, the quantity $\|\tilde{u}\|_{Z(I_j)} \le \delta$ is suitably small with δ to be chosen later. The integral equation of w with initial time t_j is

$$w(t) = e^{i(t-t_j)\Delta^2}w(t_j) - i\int_{t_j}^{t} e^{i(t-s)\Delta^2}[|w+\tilde{u}|^{p-1}(w+\tilde{u}) - |\tilde{u}|^{p-1}\tilde{u} - e](s)ds. \tag{2.10}$$

Using the inhomogeneous Strichartz estimates (2.7) on I_j, we obtain

$$\begin{aligned}\|w\|_{Z(I_j)} &\le \|e^{i(t-t_j)\Delta^2}w(t_j)\|_{Z(I_j)} + c\||w+\tilde{u}|^{p-1}(w+\tilde{u}) + |\tilde{u}|^{p-1}\tilde{u}\|_{Z'(I_j)} + \|e\|_{L^{q'}(I;L^{r'})}\\ &\le \|e^{i(t-t_j)\Delta^2}w(t_j)\|_{Z(I_j)} + c\|\tilde{u}\|^{p-1}_{Z(I_j)}\|w\|_{Z(I_j)} + c\|w\|^p_{Z(I_j)} + \|e\|_{L^{q'}(I;L^{r'})}\\ &\le \|e^{i(t-t_j)\Delta^2}w(t_j)\|_{Z(I_j)} + c\delta^{p-1}\|w\|_{Z(I_j)} + c\|w\|^p_{Z(I_j)} + c\epsilon_0.\end{aligned}$$

If

$$\delta \le \left(\frac{1}{4c}\right)^{\frac{1}{p-1}}, \qquad \|e^{i(t-t_j)\Delta^2}w(t_j)\|_{Z(I_j)} + c\epsilon_0 \le \frac{1}{2}\left(\frac{1}{4c}\right)^{\frac{1}{p-1}}, \tag{2.11}$$

then

$$\|w\|_{Z(I_j)} \le 2\|e^{i(t-t_j)\Delta^2}w(t_j)\|_{Z(I_j)} + c\epsilon_0.$$

Now take $t = t_{j+1}$ in (2.12), and apply $e^{i(t-t_{j+1})\Delta^2}$ to both sides to obtain

$$e^{i(t-t_{j+1})\Delta^2}w(t_{j+1}) = e^{i(t-t_j)\Delta^2}w(t_j) - i\int_{t_j}^{t_{j+1}} e^{i(t-s)\Delta^2}[|w+\tilde{u}|^{p-1}(w+\tilde{u}) - |\tilde{u}|^{p-1}\tilde{u} - e](s)ds. \tag{2.12}$$

Since the Duhamel integral is confined to I_j, using the inhomogeneous Strichart'z estimates (2.7) and following a similar argument as above, we obtain that

$$\begin{aligned}\|e^{i(t-t_{j+1})\Delta^2}w(t_{j+1})\|_{Z(I_+)} &\le \|e^{i(t-t_j)\Delta^2}w(t_j)\|_{Z(I_+)} + c\delta^{p-1}\|w\|_{Z(I_j)} + c\|w\|^p_{Z(I_j)} + c\epsilon_0\\ &\le 2\|e^{i(t-t_j)\Delta^2}w(t_j)\|_{Z(I_+)} + c\epsilon_0.\end{aligned}$$

Iterating beginning with $j = 0$, we obtain

$$\|e^{i(t-t_j)\Delta^2}w(t_j)\|_{Z(I_+)} \le 2^j\|e^{i(t-t_0)\Delta^2}w(t_0)\|_{Z(I_+)} + (2^j-1)c\epsilon_0 \le 2^{j+2}c\epsilon_0.$$

To accommodate the conditions (2.11) for all intervals I_j with $0 \le j \le N-1$, we require

$$2^{N+2} c\epsilon_0 \le \left(\frac{1}{4c}\right)^{\frac{1}{p-1}}. \tag{2.13}$$

Finally,

$$\|w\|_{Z(I_+)} \le \sum_{j=0}^{N-1} 2^{j+2} c\epsilon_0 + cN\epsilon_0 \le c(N)\epsilon_0,$$

which implies $\|w\|_{Z(I_+)} \le c(A)\epsilon_0$ since $N = N(A)$, concluding the proof.

□

3. Variational Structure

Following the idea from [34], we study the variational structure of the ground state of the elliptic equation (1.3) by seeking the best constant of the Gagliardo-Nirenberg inequality

$$\|u\|_{p+1}^{p+1} \le C_{GN} \|u\|_2^{p+1-\frac{N(p-1)}{4}} \|\Delta u\|_2^{\frac{N(p-1)}{4}}. \tag{3.1}$$

Formally, if W is the minimizer of the variational problem

$$J = \inf\{J(u) : u \in H^2\} \quad \textit{with} \quad J(u) = \frac{\|u\|_2^{p+1-\frac{N(p-1)}{4}} \|\Delta u\|_2^{\frac{N(p-1)}{4}}}{\|u\|_{p+1}^{p+1}}, \tag{3.2}$$

then we compute straightforward to get that W satisfies the equation

$$\begin{aligned} &\|W\|_2^{p-1-\frac{N(p-1)}{4}} \|\Delta W\|_2^{\frac{N(p-1)}{4}} (1 + \frac{(4-N)(p-1)}{8}) W \\ &+ \|W\|_2^{p+1-\frac{N(p-1)}{4}} \|\Delta W\|_2^{\frac{N(p-1)}{4}-2} \frac{N(p-1)}{8} \Delta^2 W - J^{\frac{p+1}{2}} |W|^{p-1} W = 0. \end{aligned}$$

If we set $W(x) = aQ(x)$, where a, b satisfies

$$\frac{N(p-1)}{8} b^4 (2 - s_c)(1 + \frac{(4-N)(p-1)}{8})^{-1} = 1$$

and

$$J^{\frac{p+1}{2}} a^{p-1} (2 - s_c)(1 + \frac{(4-N)(p-1)}{8})^{-1} = 1,$$

then $Q(x) = a^{-1} W(b^{-1} x)$ solves the equation (1.3), and also attains the variational problem (3.2) with $J = J(Q) = \frac{\|Q\|_2^{p+1-\frac{N(p-1)}{4}} \|\Delta Q\|_2^{\frac{N(p-1)}{4}}}{\|Q\|_{p+1}^{p+1}}$ (noting that $J(Q) = J(W)$ is invariant under the scaling $Q(x) = a^{-1} W(b^{-1} x)$).

The existence of the ground state solution of (1.3) can be shown by the same method as used in [33, 2], so we omit here. Moreover, the Pohozeav identity

$$(2 - \frac{N}{2}) \|\Delta Q\|_2^2 - (2 - s_c) \frac{N}{2} \|Q\|_2^2 + \frac{N}{p+1} \|Q\|_{p+1}^{p+1} = 0,$$

which can be obtained by multiplying the equation (1.3) by $x \cdot \nabla Q$, combined with the identity $\|\Delta Q\|_2^2 + (2 - s_c)\|Q\|_2^2 - \|Q\|_{p+1}^{p+1} = 0$, obtained by multiplying the equation (1.3) by Q, implies immediately that

$$\|\Delta Q\|_2^2 = \frac{N(p-1)}{4(p+1)}\|Q\|_{p+1}^{p+1}, \quad \|Q\|_2^2 = \frac{p-1}{2(p+1)}\|Q\|_{p+1}^{p+1}, \quad E(Q) = \frac{N(p-1)-8}{8(p+1)}\|Q\|_{p+1}^{p+1}. \tag{3.3}$$

Then we have

$$C_{GN} = \frac{1}{J} = \frac{4(p+1)}{N(p-1)} \frac{1}{\|Q\|_2^{p+1-\frac{N(p-1)}{4}} \|\Delta Q\|_2^{\frac{N(p-1)}{4}-2}}. \tag{3.4}$$

4. Global versus blow-up and dichotomy

Theorem 4.1. *Let $u_0 \in H^2$ and $I = (T_-, T_+)$ be the maximal time interval of existence of $u(t)$ solving* (1.2)*. Suppose that*

$$M(u)^{\frac{2-s_c}{s_c}} E(u) < M(Q)^{\frac{2-s_c}{s_c}} E(Q). \tag{4.1}$$

If (4.1) *holds and*

$$\|u_0\|_2^{\frac{2-s_c}{s_c}} \|\Delta u_0\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}} \|\Delta Q\|_2, \tag{4.2}$$

then $I = (-\infty, +\infty)$, i.e., the solution exists globally in time, and for all time $t \in \mathbb{R}$,

$$\|u(t)\|_2^{\frac{2-s_c}{s_c}} \|\Delta u(t)\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}} \|\Delta Q\|_2. \tag{4.3}$$

If (4.1) *holds and*

$$\|u_0\|_2^{\frac{2-s_c}{s_c}} \|\Delta u_0\|_2 > \|Q\|_2^{\frac{2-s_c}{s_c}} \|\Delta Q\|_2, \tag{4.4}$$

then for $t \in I$,

$$\|u(t)\|_2^{\frac{2-s_c}{s_c}} \|\Delta u(t)\|_2 > \|Q\|_2^{\frac{2-s_c}{s_c}} \|\Delta Q\|_2. \tag{4.5}$$

Proof. Multiplying the definition of energy by $M(u)^{\frac{2-s_c}{s_c}}$ and using (3.1), we have

$$\begin{aligned} M(u)^{\frac{2-s_c}{s_c}} E(u) =& \frac{1}{2}\|u(t)\|_2^{\frac{2(2-s_c)}{s_c}} \|\Delta u(t)\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}\|u\|_2^{\frac{2(2-s_c)}{s_c}} \\ \geq& \frac{1}{2}(\|u(t)\|_2^{\frac{2-s_c}{s_c}} \|\Delta u(t)\|_2)^2 - \frac{C_{GN}}{p+1}(\|u(t)\|_2^{\frac{2-s_c}{s_c}} \|\Delta u(t)\|_2)^{\frac{N(p-1)}{4}}. \end{aligned}$$

Define $f(x) = \frac{1}{2}x^2 - \frac{1}{p+1}C_{GN}x^{\frac{N(p+1)}{4}}$. Then $f'(x) = x\left(1 - C_{GN}\frac{N(p-1)}{4(p+1)}x^{\frac{N(p+1)-8}{4}}\right)$, and thus, $f'(x) = 0$ when $x_0 = 0$ and $x_1 = \|Q\|_2^{\frac{2-s_c}{s_c}} \|\Delta Q\|_2$. The graph of f has a local minimum at x_0 and a local maximum at x_1. The condition (4.1) and (3.3) imply that $M(u_0)^{\frac{2-s_c}{s_c}} E(u_0) < f(x_1) = M(Q)^{\frac{2-s_c}{s_c}} E(Q)$. This combined with energy conservation gives that

$$f(\|u(t)\|_2^{\frac{2-s_c}{s_c}} \|\Delta u(t)\|_2) \leq M(u(t))^{\frac{2-s_c}{s_c}} E(u(t)) = M(u_0)^{\frac{2-s_c}{s_c}} E(u_0) < f(x_1). \tag{4.6}$$

If initially $\|u_0\|_2^{\frac{2-s_c}{s_c}}\|\Delta u_0\|_2 < x_1$, then by (4.6) and the continuity of $\|\Delta u(t)\|_2$ in t, we have (4.3) for all time $t \in I$. In particular, the H^2-norm of the solution u is bounded, which, by Proposition 2.1, proves the global existence in this case. If initially $\|u_0\|_2^{\frac{2-s_c}{s_c}}\|\Delta u_0\|_2 > x_1$, then by (4.6) and the continuity of $\|\Delta u(t)\|_2$ in t, we have (4.5) for all time $t \in I$.

From the argument above, we can refine this analysis to obtain the following. If the condition (4.2) holds, then there exists $\delta > 0$ such that $M(u)^{\frac{2-s_c}{s_c}}E(u) < (1-\delta)M(Q)^{\frac{2-s_c}{s_c}}E(Q)$, and thus there exists $\delta_0 = \delta_0(\delta)$ such that $\|u(t)\|_2^{\frac{2-s_c}{s_c}}\|\Delta u(t)\|_2 < (1-\delta_0)\|Q\|_2^{\frac{2-s_c}{s_c}}\|\Delta Q\|_2$. □

The next two lemmas provide some additional estimates under the hypotheses (4.1) and (4.2) in Theorem 4.1. These lemmas will be needed in the proof of Theorem 1.1 through a virial-type estimate, which will be established in the last two sections.

Lemma 4.2. *Let $u_0 \in H^2$ satisfy* (4.1) *and* (4.2)*. Furthermore, take $\delta > 0$ such that $M(u_0)^{\frac{2-s_c}{s_c}}E(u_0) < (1-\delta)M(Q)^{\frac{2-s_c}{s_c}}E(Q)$. If u is a solution of problem* (1.2) *with initial data u_0, then there exists $C_\delta > 0$ such that for all $t \in \mathbb{R}$,*

$$\|\Delta u\|_2^2 - \frac{N(p-1)}{4(p+1)}\|u\|_{p+1}^{p+1} \geq C_\delta \|\Delta u\|_2^2. \tag{4.7}$$

Proof. By the analysis in the proof of Theorem 4.1, there exists $\delta_0 = \delta_0(\delta) > 0$ such that for all $t \in \mathbb{R}$,

$$\|u(t)\|_2^{\frac{2-s_c}{s_c}}\|\Delta u(t)\|_2 < (1-\delta_0)\|Q\|_2^{\frac{2-s_c}{s_c}}\|\Delta Q\|_2. \tag{4.8}$$

Let

$$h(t) = \frac{1}{\|Q\|_2^{\frac{2(2-s_c)}{s_c}}\|\Delta Q\|_2^2}\left(\|u(t)\|_2^{\frac{2(2-s_c)}{s_c}}\|\Delta u(t)\|_2^2 - \frac{N(p-1)}{4(p+1)}\|u\|_{p+1}^{p+1}\|u(t)\|_2^{\frac{2(2-s_c)}{s_c}}\right)$$

and set $g(y) = y^2 - y^{\frac{N(p-1)}{4}}$. By Gagliardo-Nirenberg estimate (3.1) with sharp constant C_{GN} (3.4), we can obtain

$$h(t) \geq g\left(\frac{\|u(t)\|_2^{\frac{2-s_c}{s_c}}\|\Delta u(t)\|_2}{\|Q\|_2^{\frac{2-s_c}{s_c}}\|\Delta Q\|_2}\right).$$

By (4.8), we restrict our attention to $0 \leq y \leq 1-\delta_0$. The elementary argument gives a constant C_δ such that $g(y) \geq C_\delta y^2$ if $0 \leq y \leq 1-\delta_0$. This indeed implies (4.7). □

Lemma 4.3. *(Comparability of the kinetic energy and the total energy) Let $u_0 \in H^2$ satisfy* (4.1) *and* (4.2)*. Then*

$$\frac{N(p-1)-8}{2N(p-1)}\|\Delta u(t)\|_2^2 \leq E(u) \leq \frac{1}{2}\|\Delta u(t)\|_2^2.$$

Proof. The expression of $E(u)$ gives the second inequality immediately. The first one can be obtained from

$$\frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \geq \frac{1}{2}\|\Delta u\|_2^2 (1 - \frac{2C_{GN}}{p+1}\|\Delta u\|_2^{\frac{N(p+1)}{4}-2}\|u\|_2^{p+1-\frac{N(p+1)}{4}})$$

$$= \frac{1}{2}\|\Delta u\|_2^2 \left(1 - \frac{8}{N(p-1)}\left(\frac{\|u(t)\|_2^{\frac{2-s_c}{s_c}}\|\Delta u(t)\|_2}{\|Q\|_2^{\frac{2-s_c}{s_c}}\|\Delta Q\|_2}\right)^{\frac{N(p-1)-8}{4}}\right) \geq \frac{N(p-1)-8}{2N(p-1)}\|\Delta u\|_{L^2}^2,$$

where we have used (3.1) and (3.3).

□

To establish the scattering theory, we need the following result.

Proposition 4.4. *(Existence of wave operators) Suppose $\psi^+ \in H^2$ and*

$$\frac{1}{2}\|\psi^+\|^{\frac{2(2-s_c)}{s_c}}\|\Delta\psi^+\|_2^2 < E(Q)M(Q)^{\frac{2-s_c}{s_c}}. \tag{4.9}$$

Then there exists $v_0 \in H^2$ such that the solution v of (1.2) *with initial data v_0 satisfies*

$$\|\Delta v(t)\|_2\|v_0\|_2^{\frac{2-s_c}{s_c}} < \|\Delta Q\|_2\|Q\|_2^{\frac{2-s_c}{s_c}}, \quad M(v) = \|\psi^+\|_2^2, \quad E(v) = \frac{1}{2}\|\Delta\psi^+\|_2^2,$$

and $\lim_{t\to+\infty}\|v(t) - e^{it\Delta}\psi^+\|_{H^2} = 0$. *Moreover, if $\|e^{it\Delta}\psi^+\|_{Z([0,\infty))} \leq \delta_{sd}$, then*

$$\|v\|_{Z([0,\infty))} \leq c\|e^{it\Delta}\psi^+\|_{Z([0,\infty))}, \quad \|D^{s_c}v\|_2 \leq c\|\psi^+\|_{H^2}.$$

A similar result holds for the case $t \to -\infty$.

Proof. Similar to the proof of the small data scattering theory Proposition 2.2, we can solve the integral equation

$$v(t) = e^{it\Delta^2}\psi^+ + i\int_t^\infty e^{i(t-s)\Delta^2}|v|^{p-1}v(s)ds \tag{4.10}$$

for $t \geq T$ with T large.

In fact, there exists some large T such that $\|e^{it\Delta}\psi^+\|_{Z([T,\infty))} \leq \delta_{sd}$, where δ_{sd} is defined by Proposition 2.2. Then, the same arguments as used in Proposition 2.2 give a solution $v \in C([T,\infty), H^2)$ of (4.10). Moreover, we also have $\|v\|_{Z([T,\infty))} \leq 2\delta_{sd}$, $\|v\|_{L^{2(p-1)}([T,\infty),L^{\frac{N(p-1)}{2}})} \leq 2\delta_{sd}$, and $\|\Delta v\|_{L^\infty([T,\infty);L^2)} \leq c\|\Delta v_0\|_2$. Thus from

$$\|\Delta(v - e^{it\Delta^2}\psi^+)\|_{L^\infty([T,\infty);L^2)} \leq c\|v\|^{p-1}_{L^{2(p-1)}([T,\infty),L^{\frac{N(p-1)}{2}})}\|\Delta v\|_{L^\infty([T,\infty),L^2)},$$

we get that

$$\|\Delta(v - e^{it\Delta^2}\psi^+)\|_{L^\infty([T,\infty);L^2)} \to 0 \;\; \text{as } T \to \infty,$$

which implies $v(t) - e^{it\Delta^2}\psi^+ \to 0$ in H^2 as $t \to +\infty$. Thus $M(v) = \|\psi^+\|_2^2$.

Since $e^{it\Delta^2}\psi^+ \to 0$ in L^r as $t \to +\infty$ for any $r \in (2, \frac{2N}{N-4})$, we get easily that $\|e^{it\Delta^2}\psi^+\|_{p+1} \to 0$. This together with the fact that $\|\Delta e^{it\Delta^2}\psi^+\|_2$ is conserved implies

$$E(v) = \lim_{t\to\infty}\left(\frac{1}{2}\|\Delta e^{it\Delta^2}\psi^+\|_2^2 - \frac{1}{p+1}\|e^{it\Delta^2}\psi^+\|_{p+1}^{p+1}\right) = \frac{1}{2}\|\Delta\psi^+\|_2^2.$$

In view of (4.9) we immediately obtain $M(v)^{\frac{2-s_c}{s_c}}E(v) < E(Q)M(Q)^{\frac{2-s_c}{s_c}}$. Note that

$$\begin{aligned}&\lim_{t\to\infty}\|v(t)\|_2^{\frac{2(2-s_c)}{s_c}}\|\Delta v(t)\|_2^2 = \lim_{t\to\infty}\|e^{it\Delta}\psi^+\|_2^{\frac{2(2-s_c)}{s_c}}\|\Delta e^{it\Delta}\psi^+\|_2^{\frac{2(2-s_c)}{s_c}}\\ =&\|\psi^+\|_2^{\frac{2(2-s_c)}{s_c}}\|\Delta\psi^+\|_2^2 < 2E(Q)M(Q)^{\frac{2-s_c}{s_c}} = \frac{N(p-1)-8}{N(p-1)}\|Q\|_2^{\frac{2(2-s_c)}{s_c}}\|\Delta Q\|_2^2,\end{aligned}$$

where we have used (4.9) and (3.3) in the last two steps. Thus, due to Theorem 4.1, we can evolve $v(t)$ from T back to the initial time 0, concluding our proof.

□

5. Existence and compactness of a critical element

Definition 5.1. We say that $SC(u_0)$ holds if for $u_0 \in H^2$ satisfying $\|u_0\|_2^{\frac{2-s_c}{s_c}}\|\Delta u_0\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}}\|\Delta Q\|_2$ and $E(u_0)M(u_0)^{\frac{2-s_c}{s_c}} < E(Q)M(Q)^{\frac{2-s_c}{s_c}}$, the corresponding solution u of (1.2) with the maximal interval of existence $I = (-\infty, +\infty)$ satisfies

$$\|u\|_{Z(\mathbb{R})} < \infty. \tag{5.1}$$

We first claim that there exists $\delta > 0$ such that if $E(u)M(u)^{\frac{2-s_c}{s_c}} < \delta$ and $\|u_0\|_2^{\frac{2-s_c}{s_c}}\|\Delta u_0\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}}\|\Delta Q\|_2$, then (5.1) holds. In fact, by the definition of norm $\|\cdot\|_{Z(I)}$, the B-Strichartz estimate (2.2) and Lemma 4.3, we have

$$\|e^{it\Delta^2}u_0\|_{Z(\mathbb{R})}^{\frac{2}{s_c}} \le c\|u_0\|_{\dot H^{s_c}}^{\frac{2}{s_c}} \le c\|u_0\|_2^{\frac{2(2-s_c)}{s_c}}\|\Delta u_0\|_2^2 \le \frac{2N(p-1)c}{N(p-1)-8}E(u)M(u)^{\frac{2-s_c}{s_c}}.$$

So if $E(u)M(u)^{\frac{2-s_c}{s_c}} < \frac{N(p-1)-8}{2N(p-1)c}\delta_{sd}^{\frac{2}{s_c}}$, we get that $\|e^{it\Delta^2}u_0\|_{Z(\mathbb{R})} \le \delta_{sd}$. Then from Proposition 2.2, we get that $SC(u_0)$ holds, and the claim holds for $\delta = \frac{N(p-1)-8}{2N(p-1)c}\delta_{sd}^{\frac{2}{s_c}}$. Now for each δ, we define the set S_δ to be the collection of all such initial data in H^2 :

$$S_\delta = \{u_0 \in H^2 : \quad E(u)M(u)^{\frac{2-s_c}{s_c}} < \delta \quad and \quad \|u_0\|_2^{\frac{2-s_c}{s_c}}\|\Delta u_0\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}}\|\Delta Q\|_2\}.$$

We also define $(M^{\frac{2-s_c}{s_c}}E)_c = \sup\{\delta : \quad u_0 \in S_\delta \Rightarrow SC(u_0) \quad holds\}$. If $(M^{\frac{2-s_c}{s_c}}E)_c = M(Q)^{\frac{2-s_c}{s_c}}E(Q)$, then we are done. Thus we assume now

$$(M^{\frac{2-s_c}{s_c}}E)_c < M(Q)^{\frac{2-s_c}{s_c}}E(Q). \tag{5.2}$$

Remark 5.2. By the definition of $(M^{\frac{2-s_c}{s_c}}E)_c$, we can find a sequence of solutions u_n of (1.2) with initial data $u_{n,0} \in H^2$, which we rescale to satisfy $\|u_{n,0}\|_2 = 1$, such that

$\|\Delta u_{n,0}\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}} \|\Delta Q\|_2$ and $E(u_n) \downarrow (M^{\frac{2-s_c}{s_c}} E)_c$ as $n \to \infty$, and $SC(u_{n,0})$ does not hold for any n.

Our goal in this section is to show the existence of an H^2 solution u_c of (1.2) with the initial data $u_{c,0}$ such that $\|u_{c,0}\|_2^{\frac{2-s_c}{s_c}} \|\Delta u_{c,0}\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}} \|\Delta Q\|_2$, $M(u_c)^{\frac{2-s_c}{s_c}} E(u_c) = (M^{\frac{2-s_c}{s_c}} E)_c$ and $SC(u_{c,0})$ does not hold. Moreover, we show that if $\|u_c\|_{Z([0,+\infty))} = \infty$, then $K = \{u_c(x,t) | 0 \le t < \infty\}$ is precompact in H^2, and a corresponding conclusion is reached if $\|u_c\|_{Z((-\infty,0])} = \infty$.

Prior to fulfilling our main task, we first establish a profile decomposition lemma using the concentration compactness principle in the spirit of Keraani [19] and Merle [18]. We also refer to [9] for a similar result shown for the 3D cubic Schrödinger equation and to [12] for the linear profile decomposition for the one-dimensional fourth-order Schröinger equation.

Lemma 5.3. *(Profile decomposition). Let $\phi_n(x)$ be a radial uniformly bounded sequence in H^2. Then for each M there exists a subsequence of ϕ_n, which is denoted by itself, such that the following statements hold.*
(1) For each $1 \le j \le M$, there exists (fixed in n) a radial profile $\psi^j(x)$ in H^2 and a sequence (in n) of time shifts t_n^j, and there exists a sequence (in n) of remainders $W_n^M(x)$ in H^2 such that

$$\phi_n(x) = \sum_{j=1}^{M} e^{-it_n^j \Delta^2} \psi^j(x) + W_n^M(x).$$

(2) The time sequences have a pairwise divergence property, i.e., for $1 \le j \ne k \le M$,

$$\lim_{n\to+\infty} |t_n^j - t_n^k| = +\infty. \tag{5.3}$$

(3) The remainder sequence has the following asymptotic smallness property:

$$\lim_{M\to+\infty} [\lim_{n\to+\infty} \|e^{it\Delta^2} W_n^M\|_{Z(\mathbb{R})}] = 0. \tag{5.4}$$

(4) For each fixed M and any $0 \le s \le 2$, we have the asymptotic Pythagorean expansion as follows

$$\|\phi_n\|_{\dot H^s}^2 = \sum_{j=1}^{M} \|\psi^j\|_{\dot H^s}^2 + \|W_n^M\|_{\dot H^s}^2 + o_n(1), \tag{5.5}$$

where $o_n(1) \to 0$ as $n \to +\infty$.

Proof. Let c_1 be such that $\|\phi_n\|_{H^2} \le c_1$. By the definition of the norm $\|\cdot\|_{Z(I)}$, there holds the interpolation inequality

$$\|v\|_{Z(\mathbb{R})} \le \|v\|_{L^q(\mathbb{R};L^r)}^{1-\theta} \|v\|_{L^\infty(\mathbb{R};L^{\frac{2N}{N-2s_c}})}^{\theta}$$

with some $(q,r) \in \Lambda_{s_c}$ and some $\theta \in (0,1)$. This combined with the Strichartz estimates gives

$$\|e^{it\Delta} W_n^M\|_{Z(\mathbb{R})} \le c \|W_n^M\|_{\dot H^{s_c}}^{1-\theta} \|e^{it\Delta} W_n^M\|_{L^\infty(\mathbb{R};L^{\frac{2N}{N-2s_c}})}^{\theta}.$$

Since $\|W_n^M\|_{\dot{H}^{s_c}} \le c_1$, it suffices to show that

$$\lim_{M\to+\infty}[\lim_{n\to+\infty}\|e^{it\Delta^2}W_n^M\|_{L^\infty(\mathbb{R};L^{\frac{2N}{N-2s_c}})}]=0. \tag{5.6}$$

Let $A_1 = \limsup_{n\to\infty}\|e^{it\Delta^2}\phi_n\|_{L^\infty(\mathbb{R};L^{\frac{2N}{N-2s_c}})}$. If $A_1 = 0$, the proof is complete with $\psi^j = 0$ for all $1 \le j \le M$. Suppose $A_1 > 0$. Passing to a subsequence, we may assume that $\lim_{n\to\infty}\|e^{it\Delta^2}\phi_n\|_{L^\infty(\mathbb{R};L^{\frac{2N}{N-2s_c}})} = A_1$. We will show that there is a time sequence t_n^1 and a profile $\psi^1 \in H^2$ such that $e^{it_n^1\Delta^2}\phi_n \rightharpoonup \psi^1$ and

$$\|\psi^1\|_{\dot{H}^{s_c}} \ge KA_1^{\frac{N}{2s_c}+\frac{N-2s_c}{\min\{2s_c,4-2s_c\}}}\left(\frac{1}{c_1}\right)^{\frac{N}{2s_c}-1+\frac{N-2s_c}{\min\{2s_c,4-2s_c\}}}. \tag{5.7}$$

For $r > 1$ to be chosen, let $\chi(x)$ be a radial Schwartz function such that $\hat{\chi}(\xi) = 1$ for $\frac{1}{r} \le |\xi| \le r$ and $\hat{\chi}(\xi)$ is supported in $\frac{1}{2r} \le |\xi| \le 2r$.

Since the operator $e^{it\Delta^2}$ is an isometry on $\dot{H}^{s_c}$, then by the Sobolev embedding,

$$\begin{aligned}
&\|e^{it\Delta^2}\phi_n - \chi e^{it\Delta^2}\phi_n\|^2_{L^\infty(\mathbb{R};L^{\frac{2N}{N-2s_c}})} \le \sup_t\|e^{it\Delta^2}\phi_n - \chi e^{it\Delta^2}\phi_n\|^2_{\dot{H}^{s_c}}\\
\le&\sup_t\int|\xi|^{2s_c}(1-\hat{\chi}(\xi))^2|\hat{\phi}_n(\xi)|^2d\xi \le \int_{|\xi|\le\frac{1}{r}}|\xi|^{2s_c}|\hat{\phi}_n(\xi)|^2d\xi + \int_{|\xi|\ge r}|\xi|^{2s_c}|\hat{\phi}_n(\xi)|^2d\xi\\
\le&\frac{1}{r^{2s_c}}\|\phi_n\|_2 + \frac{1}{r^{4-2s_c}}\|\phi_n\|^2_{\dot{H}^2} \le (\frac{1}{r^{2s_c}}+\frac{1}{r^{4-2s_c}})c_1^2.
\end{aligned}$$

Take r sufficiently large such that $(\frac{1}{r^{2s_c}}+\frac{1}{r^{4-2s_c}})c_1^2 = \frac{A_1^2}{4}\epsilon_0$ with some $0<\epsilon_0<1$. (This implies that $\frac{1}{r} \ge c\left(\frac{A_1^2}{c_1^2}\right)^{\frac{1}{min\{2s_c,4-2s_c\}}}$.) Then, for n large, we have $\|\chi * e^{it\Delta^2}\phi_n\|_{L^\infty(\mathbb{R};L^{\frac{2N}{N-2s_c}})} \ge \frac{1}{2}A_1$. Thus from

$$\|\chi * e^{it\Delta^2}\phi_n\|^{\frac{2N}{N-2s_c}}_{L^\infty(\mathbb{R};L^{\frac{2N}{N-2s_c}})} \le \|\chi * e^{it\Delta^2}\phi_n\|^2_{L^\infty(\mathbb{R};L^2)}\|\chi * e^{it\Delta^2}\phi_n\|^{\frac{4s_c}{N-s_c}}_{L^\infty(\mathbb{R};L^\infty)} \le \|\phi_n\|_2^2\|\chi * e^{it\Delta^2}\phi_n\|^{\frac{4s_c}{N-s_c}}_{L^\infty(\mathbb{R};L^\infty)},$$

we get that $\|\chi * e^{it\Delta^2}\phi_n\|_{L^\infty(\mathbb{R};L^\infty)} \ge \left(\frac{A_1}{2}\right)^{\frac{N}{2s_c}}\left(\frac{1}{c_1^2}\right)^{\frac{N-s_c}{4s_c}}$. Since ϕ_n are radial functions, so are $\chi * e^{it\Delta^2}\phi_n$. By the radial Gagliardo-Nirenberg inequality, we obtain that

$$\|\chi * e^{it\Delta^2}\phi_n\|_{L^\infty(\mathbb{R};L^\infty(|x|\ge R))} \le \frac{1}{R}\|\chi * e^{it\Delta^2}\phi_n\|^{\frac{1}{2}}_{L^\infty(\mathbb{R};L^2)}\|\nabla\chi * e^{it\Delta^2}\phi_n\|^{\frac{1}{2}}_{L^\infty(\mathbb{R};L^2)} \le \frac{c_1}{R}.$$

Therefore, by selecting R large enough, $\|\chi * e^{it\Delta^2}\phi_n\|_{L^\infty(\mathbb{R};L^\infty(|x|\le R))} \ge \frac{1}{2}\left(\frac{A_1}{2}\right)^{\frac{N}{2s_c}}\left(\frac{1}{c_1^2}\right)^{\frac{N-s_c}{4s_c}}$. Let t_n^1 and x_n^1 with $|x_n^1| \le R$ be the sequences such that for each n, $|\chi * e^{it_n^1\Delta^2}\phi_n(x_n^1)| \ge \frac{1}{4}\left(\frac{A_1}{2}\right)^{\frac{N}{2s_c}}\left(\frac{1}{c_1^2}\right)^{\frac{N-s_c}{4s_c}}$, or

$$\left|\int\chi(x_n^1-y)e^{it_n^1\Delta^2}\phi_n(y)dy\right| \ge \frac{1}{4}\left(\frac{A_1}{2}\right)^{\frac{N}{2s_c}}\left(\frac{1}{c_1^2}\right)^{\frac{N-s_c}{4s_c}}.$$

Passing to a subsequence such that $x_n^1 \to x^1$, which is possible because $|x_n^1| \le R$, we obtain that

$$\left|\int \chi(x^1-y)e^{it_n^1\Delta^2}\phi_n(y)dy\right| \ge \frac{1}{8}\left(\frac{A_1}{2}\right)^{\frac{N}{2s_c}}\left(\frac{1}{c_1^2}\right)^{\frac{N-s_c}{4s_c}}.$$

Since the sequence $e^{it_n^1\Delta^2}\phi_n$ is uniformly bounded in H^2, then we can find a radial function $\psi^1 \in H^2$ such that, up to a subsequence, $e^{it_n^1\Delta^2}\phi_n \rightharpoonup \psi^1$ weakly in H^2 with $\|\psi^1\|_{H^2} \le c_1$. Thus,

$$\left|\int \chi(x^1-y)\psi^1(y)dy\right| \ge \frac{1}{8}\left(\frac{A_1}{2}\right)^{\frac{N}{2s_c}}\left(\frac{1}{c_1^2}\right)^{\frac{N-s_c}{4s_c}}.$$

By Plancherel and Cauchy-Schwarz inequalities, $\|\chi\|_{\dot H^{-s_c}}\|\psi^1\|_{\dot H^{s_c}} \ge \frac{1}{8}\left(\frac{A_1}{2}\right)^{\frac{N}{2s_c}}\left(\frac{1}{c_1^2}\right)^{\frac{N-s_c}{4s_c}}$. Since $\|\chi\|_{\dot H^{-s_c}} \le cr^{\frac{N}{2}-s_c}$, then $\|\psi^1\|_{\dot H^{s_c}} \ge \frac{1}{8c}\left(\frac{A_1}{2}\right)^{\frac{N}{2s_c}}\left(\frac{1}{c_1^2}\right)^{\frac{N-s_c}{4s_c}} r^{-(\frac{N}{2}-s_c)}$. In view of the choice of r, we obtain for some constant K such that (5.7) holds, concluding the claim.

Let $W_n^1 = \phi_n - e^{-it_n^1\Delta^2}\psi^1$. Then since $e^{it_n^1\Delta^2}W_n^1 \rightharpoonup 0$ weakly in H^2, for any $s \in [0,2]$,

$$< \phi_n, e^{-it_n^1\Delta^2}\psi^1 >_{\dot H^s} = < e^{it_n^1\Delta^2}\phi_n, \psi^1 >_{\dot H^s} \to \|\psi^1\|^2_{\dot H^s}.$$

By expanding $\|W_n^1\|^2_{\dot H^s}$, we obtain

$$\lim_{n\to\infty}\|W_n^1\|^2_{\dot H^s} = \lim_{n\to\infty}\|\phi_n\|^2_{\dot H^s} - \|\psi^1\|_{\dot H^s}.$$

From this with $s=0$ and $s=2$, we deduce that $\|W_n^1\|_{H^2} \le c_1$.

Let $A_2 = \limsup_{n\to\infty}\|e^{it\Delta^2}W_n^1\|_{L^\infty(\mathbb{R};L^{\frac{2N}{N-2s_c}})}$. If $A_2 = 0$, then we are done. If $A_2 > 0$, then we repeat the above argument with ϕ_n replaced by W_n^1 to obtain a sequence of time shifts t_n^2 and a profile $\psi^2 \in H^2$ such that $e^{it_n^2\Delta^2}W_n^1 \rightharpoonup \psi^2$ weakly in H^2, and

$$\|\psi^2\|_{\dot H^{s_c}} \ge K\left(A_2\right)^{\frac{N}{2s_c}+\frac{N-2s_c}{min\{2s_c,4-2s_c\}}}\left(\frac{1}{c_1}\right)^{\frac{N-s_c}{2s_c}+\frac{N-2s_c}{min\{2s_c,4-2s_c\}}}.$$

We show that $|t_n^1 - t_n^2| \to \infty$. Indeed, if we suppose, up to a subsequence, $t_n^2 - t_n^1 \to t_0$ finite, then

$$e^{i(t_n^2-t_n^1)\Delta^2}(e^{it_n^1\Delta^2}\phi_n - \psi^1) = e^{it_n^2\Delta^2}(\phi_n - e^{-it_n^1\Delta^2}\psi^1) = e^{it_n^2\Delta^2}W_n^1 \rightharpoonup \psi^2.$$

Since $e^{it_n^1\Delta^2}\phi_n - \psi^1 \rightharpoonup 0$, the left side of the above expression converges weakly to 0, and so $\psi^2 = 0$, a contradiction. Let $W_n^2 = \phi_n - e^{-it_n^1\Delta^2}\psi^1 - e^{-it_n^2\Delta^2}\psi^2$. Note that for any $s \in [0,2]$,

$$< \phi_n, e^{-it_n^2\Delta^2}\psi^2 >_{\dot H^s} = < e^{it_n^2\Delta^2}\phi_n, \psi^2 >_{\dot H^s} = < e^{it_n^2\Delta^2}(\phi_n - e^{it_n^1\Delta^2}\psi^1), \psi^2 >_{\dot H^s} + o_n(1) \to \|\psi^2\|^2_{\dot H^s}.$$

We expand

$$\lim_{n\to\infty}\|W_n^2\|^2_{\dot H^s} = \lim_{n\to\infty}\|\phi_n\|^2_{\dot H^s} - \|\psi^1\|^2_{\dot H^s} - \|\psi^2\|^2_{\dot H^s},$$

and obtain $\|W_n^2\|_{H^2} \le c_1$.

We continue inductively, constructing a sequence t_n^M and a profile ψ^M such that $e^{it_n^M\Delta^2}W_n^{M-1} \rightharpoonup \psi^M$ weakly in H^2, and

$$\|\psi^M\|_{\dot{H}^{s_c}} \geq K\left(A_M\right)^{\frac{N}{2s_c}+\frac{N-2s_c}{min\{2s_c,4-2s_c\}}}\left(\frac{1}{c_1}\right)^{\frac{N-s_c}{2s_c}+\frac{N-2s_c}{min\{2s_c,4-2s_c\}}}.$$

Suppose $1\leq j<M$. We show that $|t_n^M-t_n^j|\to\infty$ inductively by assuming that $|t_n^M-t_n^{j+1}|\to\infty,\dots,|t_n^M-t_n^{M-1}|\to\infty$. In fact, Suppose up to a subsequence $t_n^M-t_n^j\to t_0$ finite. We have

$$\begin{aligned}&e^{i(t_n^M-t_n^j)\Delta^2}(e^{it_n^j\Delta^2}W_n^{j-1}-\psi^j)-e^{i(t_n^M-t_n^{j+1})\Delta^2}\psi^{j+1}-\cdots-e^{i(t_n^M-t_n^{M-1})\Delta^2}\psi^{M-1}\\&=e^{it_n^2\Delta^2}(\phi_n-e^{it_n^1\Delta^2}\psi^1-\cdots-e^{-it_n^{M-1}\Delta^2}\psi^{M-1})=e^{it_n^M\Delta^2}W_n^{M-1}\rightharpoonup\psi^M.\end{aligned}$$

Since the left side converges weakly to 0, then we get a contradiction since $\psi^M\neq 0$. This proves (5.3). Let $W_n^M=\phi_n-e^{-it_n^1\Delta^2}\psi^1-e^{-it_n^2\Delta^2}\psi^2-\cdots-e^{-it_n^M\Delta^2}\psi^M$. Note that

$$\begin{aligned}&<\phi_n,e^{-it_n^M\Delta^2}\psi^M>_{\dot{H}^s}=<e^{it_n^M\Delta^2}\phi_n,\psi^M>_{\dot{H}^s}\\&=<e^{it_n^M\Delta^2}(\phi_n-e^{it_n^1\Delta^2}\psi^1-\cdots-e^{-it_n^{M-1}\Delta^2}\psi^{M-1}),\psi^M>_{\dot{H}^s}+o_n(1)\\&=<e^{it_n^M\Delta^2}W_n^{M-1},\psi^M>_{\dot{H}^s}+o_n(1)\to\|\psi^M\|_{\dot{H}^s}^2,\end{aligned}$$

where the second line follows from the pairwise divergence property (5.3). The expansion (5.5) is then shown by expanding $\|W_n^M\|_{\dot{H}^s}^2$.

By (5.5) and $\|\psi^M\|_{\dot{H}^{s_c}}\geq K\left(A_M\right)^{\frac{N}{2s_c}+\frac{N-2s_c}{min\{2s_c,4-2s_c\}}}\left(\frac{1}{c_1}\right)^{\frac{N-s_c}{2s_c}+\frac{N-2s_c}{min\{2s_c,4-2s_c\}}}$, we get that

$$\sum_{M=1}^{\infty}\left(K\left(A_M\right)^{\frac{N}{2s_c}+\frac{N-2s_c}{min\{2s_c,4-2s_c\}}}\left(\frac{1}{c_1}\right)^{\frac{N-s_c}{2s_c}+\frac{N-2s_c}{min\{2s_c,4-2s_c\}}}\right)^2\leq\lim_{n\to\infty}\|\phi_n\|_{\dot{H}^s}^2\leq c_1^2.$$

Since $N>2s_c$, then $A_M\to 0$ as $M\to\infty$, which implies (5.4). □

Lemma 5.4. *(Energy pythagorean expansion) In the situation of Lemma 5.3, we have*

$$E(\phi_n)=\sum_{j=1}^{M}E(e^{-it_n^j\Delta^2}\psi^j)+E(W_n^M)+o_n(1). \tag{5.8}$$

Proof. According to (5.5), it suffices to establish for all $M\geq 1$,

$$\|\phi_n\|_{p+1}^{p+1}=\sum_{j=1}^{M}\|e^{-it_n^j\Delta^2}\psi^j\|_{p+1}^{p+1}+\|W_n^M\|_{p+1}^{p+1}+o_n(1). \tag{5.9}$$

In fact, there are only two cases to consider. Case 1. There exists some j for which t_n^j converges to a finite number, which, without loss of generality, we assume is 0. In this case we will show that $\lim_{n\to\infty}\|W_n^M\|_{p+1}=0$ for $M>j$, $\lim_{n\to\infty}\|e^{-it_n^k\Delta^2}\psi^k\|_{p+1}=0$ for all $k\neq j$, and $\lim_{n\to\infty}\|\phi_n\|_{p+1}=\|\psi^j\|_{p+1}$, which gives (5.9). Case 2. For all j, $|t_n^j|\to\infty$. In this case we will show that $\lim_{n\to\infty}\|e^{-it_n^k\Delta^2}\psi^k\|_{p+1}=0$ for all k and $\lim_{n\to\infty}\|\phi_n\|_{p+1}=\lim_{n\to\infty}\|W_n^M\|_{p+1}$, which gives (5.9) again.

For Case 1, we infer from the proof of Lemma 5.3 that $W_n^{j-1} \rightharpoonup \psi^j$. By the compactness of the embedding $H^2_{rad} \hookrightarrow L^{p+1}$, it follows that $W_n^{j-1} \to \psi^j$ strongly in L^{p+1}. Let $k \neq j$. Then we get from (5.3) that $|t_n^k| \to \infty$. As argued in the proof of Lemma 5.3, from Sobolev embedding and the L^q spacetime decay estimate of the linear flow, we obtain that $\|e^{-it_n^k\Delta^2}\psi^k\|_{p+1} \to 0$. Recalling that

$$W_n^{j-1} = \phi_n - e^{-it_n^1\Delta^2}\psi^1 - \cdots - e^{-it_n^{j-1}\Delta^2}\psi^{j-1},$$

we conclude that $\phi_n \to \psi^j$ strongly in L^{p+1}. Since

$$W_n^M = W_n^{j-1} - \psi^j - e^{-it_n^{j+1}\Delta^2}\psi^{j+1} - \cdots - e^{-it_n^M\Delta^2}\psi^M,$$

we also conclude that $\lim_{n\to\infty}\|W_n^M\|_{p+1} \to 0$ strongly in L^{p+1}, for $M > j$.

Case 2 follows similarly from the proof of Case 1. $\square$

Proposition 5.5. *There exists a radial $u_{c,0}$ in H^2 with*

$$E(u_{c,0}) = (M^{\frac{2-s_c}{s_c}}E)_c < M(Q)^{\frac{2-s_c}{s_c}}E(Q), \quad \|\Delta u_{c,0}\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}}\|\Delta Q\|_2$$

such that if u_c is the corresponding solution of (1.2) *with the initial data $u_{c,0}$, then u_c is global and $\|u_c\|_{Z(\mathbb{R})} = +\infty$.*

Proof. Recall from Remark 5.2, we have obtained a radial sequence u_n with $\|u_n\|_2 = 1$ in the beginning of this section, satisfying $\|\Delta u_{n,0}\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}}\|\Delta Q\|_2$ and $E(u_n) \downarrow (M^{\frac{2-s_c}{s_c}}E)_c$ as $n \to \infty$. Each u_n is global and nonscattering such that $\|u_n\|_{Z(\mathbb{R})} = +\infty$. We apply Lemma 5.3 to $u_{n,0}$, which is uniformly bounded in H^2 to get

$$u_{n,0}(x) = \sum_{j=1}^{M} e^{-it_n^j\Delta^2}\psi^j(x) + W_n^M(x). \tag{5.10}$$

Then by (5.8), we have further

$$\sum_{j=1}^{M}\lim_n E(e^{-it_n^j\Delta^2}\psi^j) + \lim_n E(W_n^M) = \lim_n E(u_{n,0}) = (M^{\frac{2-s_c}{s_c}}E)_c.$$

Since also by the profile expansion, we have

$$\|\Delta u_{n,0}\|_2^2 = \sum_{j=1}^{M}\|\Delta e^{-it_n^j\Delta^2}\psi^j\|_2^2 + \|\Delta W_n^M\|_2^2 + o_n(1),$$

and

$$1 = \|u_{n,0}\|_2^2 = \sum_{j=1}^{M}\|\psi^j\|_2^2 + \|W_n^M\|_2^2 + o_n(1). \tag{5.11}$$

Since from the proof of Lemma 4.3, each energy is nonnegative and then

$$\lim_n E(e^{-it_n^j\Delta^2}\psi^j) \le (M^{\frac{2-s_c}{s_c}}E)_c. \tag{5.12}$$

Now, if more than one $\psi^j \neq 0$, we will show a contradiction and thus the profile expansion will be reduced to the case that only one profile is not null.

In fact, if more than one $\psi^j \neq 0$, then by (5.11) we must have $M(\psi^j) < 1$ for each j, which together with (5.12) implies that for n large enough,

$$M(e^{-it_n^j\Delta^2}\psi^j)^{\frac{2-s_c}{s_c}} E(e^{-it_n^j\Delta}\psi^j) < (M^{\frac{2-s_c}{s_c}}E)_c.$$

For a given j, if $|t_n^j| \to +\infty$ we assume $t_n^j \to +\infty$ or $t_n^j \to -\infty$ up to a subsequence. In this case, by the proof of Lemma 5.4, we have $\lim_{n\to+\infty} \|e^{-it_n^j\Delta}\psi^j\|_{p+1}^{p+1} = 0$, and thus, $\frac{1}{2}\|\psi^j\|_2^{\frac{2(2-s_c)}{s_c}}\|\Delta\psi^j\|_2^2 = \frac{1}{2}\|e^{-it_n^j\Delta^2}\psi^j\|_2^{\frac{2(2-s_c)}{s_c}}\|\Delta e^{-it_n^j\Delta^2}\psi^j\|_2^2 < (M^{\frac{2-s_c}{s_c}}E)_c$. If we denote by $NLS(t)\phi$ a solution of (1.2) with initial data ϕ, then we get from the existence of wave operators (Proposition 4.4)that there exists $\tilde{\psi}^j$ such that

$$\|NLS(-t_n^j)\tilde{\psi}^j - e^{-it_n^j\Delta^2}\psi^j\|_{H^2} \to 0, \quad as \quad n \to +\infty$$

with

$$\|\tilde{\psi}^j\|_2^{\frac{2-s_c}{s_c}}\|\Delta NLS(t)\tilde{\psi}^j\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}}\|\Delta Q\|_2$$
$$\|\tilde{\psi}^j\|_2 = \|\psi^j\|_2, \quad E(\tilde{\psi}^j) = \frac{1}{2}\|\Delta\psi^j\|_2^2.$$

Thus we get that $M(\tilde{\psi}^j)^{\frac{2-s_c}{s_c}}E(\tilde{\psi}^j) < (M^{\frac{2-s_c}{s_c}}E)_c$, and so $\|NLS(t)\tilde{\psi}^j\|_{Z(\mathbb{R})} < +\infty$. If, on the other hand, for the given j, $t_n^j \to t'$ finite (at most only one such j by (5.3)), then by the continuity of the linear flow in H^2, we have

$$e^{-it_n^j\Delta^2}\psi^j \to e^{-it'\Delta^2}\psi^j \quad strongly \ \ in \ \ H^2.$$

In this case, we set $\tilde{\psi}^j = NLS(t')[e^{-it'\Delta^2}\psi^j]$ so that $NLS(-t')\tilde{\psi}^j = e^{-it'\Delta^2}\psi^j$. To sum up, in either case, we obtain a new profile $\tilde{\psi}^j$ for the given ψ^j such that

$$\|NLS(-t_n^j)\tilde{\psi}^j - e^{-it_n^j\Delta^2}\psi^j\|_{H^2} \to 0, \quad as \quad n \to +\infty. \tag{5.13}$$

As a result, we can replace $e^{-it_n^j\Delta^2}\psi^j$ by $NLS(-t_n^j)\tilde{\psi}^j$ in (5.10) and obtain

$$u_{n,0}(x) = \sum_{j=1}^{M} NLS(-t_n^j)\tilde{\psi}^j(x) + \tilde{W}_n^M(x),$$

with $\lim_{M\to+\infty}[\lim_{n\to+\infty}\|e^{it\Delta^2}\tilde{W}_n^M\|_{Z(\mathbb{R})}] = 0$.

In order to use the perturbation theory to get a contradiction, we set $v^j(t) = NLS(t)\tilde{\psi}^j$, $u_n(t) = NLS(t)u_{n,0}$ and set $\tilde{u}_n(t) = \sum_{j=1}^{M} v^j(t - t_n^j)$. Then we have

$$i\partial_t\tilde{u}_n + \Delta^2\tilde{u}_n - |\tilde{u}_n|^{p-1}\tilde{u}_n = e_n,$$

where

$$e_n = \sum_{j=1}^{M} |v^j(t-t_n^j)|^{p-1}v^j(t-t_n^j) - |\sum_{j=1}^{M} v^j(t-t_n^j)|^{p-1}\sum_{j=1}^{M} v^j(t-t_n^j).$$

We will prove the following two claims to get the contradiction:

Claim 1. There exists a large constant A independent of M such that for any M, there

exists $n_0 = n_0(M)$ such that for $n > n_0$, $\|\tilde{u}_n\|_{Z(\mathbb{R})} \leq A$.
Claim 2. For each M and $\epsilon > 0$, there exists $n_1 = n_1(M, \epsilon)$ such that for $n > n_1$ $\|e_n\|_{Z'(I)} \leq \epsilon$.
Note that, since $\tilde{u}_n(0) - u_n(0) = \tilde{W}_n^M$, there exists $M_1 = M_1(\epsilon)$ such that for each $M > M_1$ there exists $n_2 = n_2(M)$ satisfying $\|e^{it\Delta^2}(\tilde{u}_n(0) - u_n(0))\|_{Z(\mathbb{R})} \leq \epsilon$ with $\epsilon < \epsilon_0$ as in Lemma 2.4. Thus, if the two claims hold true, by the long-time perturbation theory (Lemma 2.4), for n and M large enough, we obtain $\|u_n\|_{Z(\mathbb{R})} < +\infty$, which is a contradiction. So it suffices to show the above claims.

Let M_0 sufficiently large such that $\|e^{it\Delta^2}\tilde{W}_n^{M_0}\|_{Z(\mathbb{R})} \leq \frac{1}{2}\delta_{sd}$. Thus we get from the definition of $\tilde{W}_n^{M_0}$ that for each $j > M_0$, $\|e^{it\Delta^2}v^j(-t_n^j)\|_{Z(\mathbb{R})} \leq \delta_{sd}$. Similar to the small data scattering and Proposition 4.4, we obtain

$$\|v^j(t - t_n^j)\|_{Z(\mathbb{R})} \leq 2\|e^{it\Delta^2}v^j(-t_n^j)\|_{Z(\mathbb{R})} \leq 2\delta_{sd}, \quad \|D^{s_c}v^j\|_2 \leq c\|\psi^j\|_{\dot{H}^{s_c}}. \tag{5.14}$$

Thus by the elementary inequality

$$|(\sum_{j=1}^{M} a_j)^p - \sum_{j=1}^{M} a_j^p| \leq c_M \sum_{j \neq k} |a_j||a_k|^{p-1}, p > 1, a_j \geq 0$$

and (5.13), we have that

$$\begin{aligned}\|\tilde{u}_n\|_{Z(\mathbb{R})}^{\frac{(N+4)(p-1)}{4}} &= \sum_{j=1}^{M_0} \|v^j(t - t_n^j)\|_{Z(\mathbb{R})}^{\frac{(N+4)(p-1)}{4}} + \sum_{j=M_0+1}^{M} \|v^j(t - t_n^j)\|_{Z(\mathbb{R})}^{\frac{(N+4)(p-1)}{4}} + crossterms \\ &\leq \sum_{j=1}^{M_0} \|v^j\|_{Z(\mathbb{R})}^{\frac{(N+4)(p-1)}{4}} + c \sum_{j=M_0+1}^{M} \|\psi^j\|_{\dot{H}^{s_c}}^{\frac{(N+4)(p-1)}{4}} + crossterms.\end{aligned} \tag{5.15}$$

In view of (5.3), by taking n_0 large enough, the *crossterms* can be made bounded. On the other hand, by (5.10) and Lemma 5.3,

$$\|u_{n,0}\|_{\dot{H}^{s_c}}^2 = \sum_{j=1}^{M_0} \|\psi^j\|_{\dot{H}^{s_c}}^2 + \sum_{j=M_0+1}^{M} \|\psi^j\|_{\dot{H}^{s_c}}^2 + \|W_n^M\|_{\dot{H}^{s_c}}^2 + o_n(1), \tag{5.16}$$

which shows that the quantity $\sum_{j=M_0+1}^{M} \|\psi^j\|_{\dot{H}^{s_c}}^{\frac{(N+4)(p-1)}{4}}$ is bounded independently of M since $\frac{(N+4)(p-1)}{4} > \frac{N(p-1)}{4} > 2$. Above all, (5.15) gives that $\|\tilde{u}_n\|_{Z(\mathbb{R})}$ is bounded independently of M for $n > n_0$ large enough. So the first claim holds true.

On the other hand, since

$$e_n = \left(|v^j(t - t_n^j)|^{p-1} - |\sum_{k=1}^{M} v^k(t - t_n^k)|^{p-1} \right) \sum_{j=1}^{M} v^j(t - t_n^j),$$

then if $p-1>1$, we estimate

$$|e_n| \le c\sum\sum_{k\neq j}^{M} |v^j(t-t_n^j)||v^k(t-t_n^k)|\Big(|v^j(t-t_n^j)|^{p-2}+|v^k(t-t_n^k)|^{p-2}\Big);$$

while if $p-1<1$,

$$|e_n| \le c\sum\sum_{k\neq j}^{M} |v^j(t-t_n^j)||v^k(t-t_n^k)|^{p-1}.$$

Since by (5.3), for $j\neq k$, $|t_n^j-t_n^k|\to+\infty$, then we obtain that $\|e_n\|_{Z(\mathbb{R})}$ goes to zero as $n\to\infty$, concluding the second claim.

Up to now, we have reduced the profile expansion to the case that $\psi^1\neq 0$, and $\psi^j=0$ for all $j\geq 2$. We now begin to show the existence of a critical solution. By (5.11) we have $M(\psi^1)\le 1$, and by (5.12) we have $\lim_n E(e^{-it_n^1\Delta^2}\psi^1)\le (M^{\frac{2-s_c}{s_c}}E)_c$. If t_n^1 converges and, without loss of generality, $t_n^1\to 0$ as $n\to+\infty$, we take $\tilde\psi^1=\psi^1$ and then we have $\|NLS(-t_n^1)\tilde\psi^1-e^{-it_n^1\Delta^2}\psi^1\|_{H^2}\to 0$ as $n\to+\infty$. If, on the other hand, $t_n^1\to+\infty$, then by the proof of Lemma 5.4, we have $\lim_{n\to+\infty}\|e^{-it_n^1\Delta^2}\psi^1\|_{p+1}^{p+1}=0$, and so

$$\frac{1}{2}\|\Delta\psi^1\|_2^2=\lim_n E(e^{-it_n^1\Delta^2}\psi^1)\le (M^{\frac{2-s_c}{s_c}}E)_c.$$

Thus, by Proposition 4.4, there exist $\tilde\psi^1$ such that $M(\tilde\psi^1)=M(\psi^1)\le 1$, $E(\tilde\psi^1)=\frac{1}{2}\|\Delta\psi^1\|_2^2\le (M^{\frac{2-s_c}{s_c}}E)_c$, and $\|NLS(-t_n^1)\tilde\psi^1-e^{-it_n^1\Delta}\psi^1\|_{H^2}\to 0$ as $n\to+\infty$. In either case, if we set $\tilde W_n^M=W_n^M+(e^{-it_n^1\Delta^2}\psi^1-NLS(-t_n^1)\tilde\psi^1)$, then by Strichartz estimates, we have

$$\|e^{it\Delta}\tilde W_n^M\|_{Z(\mathbb{R})}\le \|e^{it\Delta^2}W_n^M\|_{Z(\mathbb{R})}+c\|e^{-it_n^1\Delta^2}\psi^1-NLS(-t_n^1)\tilde\psi^1\|_{\dot H^{s_c}},$$

and thus

$$\lim_{n\to+\infty}\|e^{it\Delta^2}\tilde W_n^M\|_{Z(\mathbb{R})}=\lim_{n\to+\infty}\|e^{it\Delta^2}W_n^M\|_{Z(\mathbb{R})}.$$

Therefore, we have $u_{n,0}=NLS(-t_n^1)\tilde\psi^1+\tilde W_n^M$ with $M(\tilde\psi^1)\le 1$, $E(\tilde\psi^1)\le (M^{\frac{2-s_c}{s_c}}E)_c$ and $\lim_{M\to+\infty}[\lim_{n\to+\infty}\|e^{it\Delta^2}\tilde W_n^M\|_{Z(\mathbb{R})}]=0$. Let u_c be the solution of (1.2) with initial data $u_{c,0}=\tilde\psi^1$. We claim that $\|u_c\|_{Z(\mathbb{R})}=\infty$, and then it must hold that $M(u_c)=1$ and $E(u_c)=(M^{\frac{2-s_c}{s_c}}E)_c$ by the definition of $(M^{\frac{2-s_c}{s_c}}E)_c$, concluding the proof. Contrarily, if otherwise $A\equiv\|NLS(t-t_n^1)\tilde\psi^1)\|_{Z(\mathbb{R})}=\|NLS(t)\tilde\psi^1)\|_{Z(\mathbb{R})}=\|u_c\|_{Z(\mathbb{R})}<\infty$. From the perturbation theory (Lemma 2.4), we get a $\epsilon_0=\epsilon_0(A)$. Taking M sufficiently large and $n_2(M)$ large enough such that for $n>n_2$, it holds that $\|e^{it\Delta^2}\tilde W_n^M\|_{Z(\mathbb{R})}\le\epsilon_0$. Similar to the proof of the first case, Lemma 2.4 implies that there exists a large n such that $\|u_n\|_{Z(\mathbb{R})}<\infty$, which is a contradiction .

□

Proposition 5.6. *(Precompactness of the flow of the critical solution) Let u_c be as in Proposition 5.5. If $\|u_c\|_{Z([0,\infty))}=\infty$, then*

$$K=\{u_c(t)|\ t\in[0,+\infty)\}\subset H^2$$

is precompact in H^2. A corresponding conclusion is reached if $\|u_c\|_{Z((-\infty,0])} = \infty$.

Proof. Take a sequence $t_n \to +\infty$. We argue that $u_c(t_n)$ has a subsequence converging in H^2. In the sequel, we write $u = u_c$ for short. Take $\phi_n = u(t_n)$ in the profile expansion lemma 5.3 to obtain profiles ψ^j and a remainder W_n^M such that

$$u(t_n) = \sum_{j=1}^{M} e^{-it_n^j \Delta^2}\psi^j + W_n^M$$

with $|t_n^j - t_n^k| \to +\infty$ as $n \to +\infty$ for any $j \neq k$. Then Lemma 5.4 gives

$$\sum_{j=1}^{M} \lim_{n\to+\infty} E(e^{-it_n^j\Delta^2}\psi^j) + \lim_{n\to+\infty} E(W_n^M) = E(u) = (M^{\frac{2-s_c}{s_c}}E)_c.$$

Similar to the proof of Lemma 4.3, we know that each energy is nonnegative and thus for any j,

$$\lim_{n\to+\infty} E(e^{-it_n^j\Delta^2}\psi^j) \leq (M^{\frac{2-s_c}{s_c}}E)_c.$$

Moreover by (5.5), we have

$$\sum_{j=1}^{M} M(\psi^j) + \lim_{n\to+\infty} M(W_n^M) = \lim_{n\to+\infty} M(u(t_n)) = 1.$$

If more than one $\psi^j \neq 0$. Similar to the proof in Proposition 5.5, we a contradiction from the definition of the critical solution $u = u_c$. Thus we will address the case that only $\psi^1 \neq 0$ and $\psi^j = 0$ for all $j > 1$, and so

$$u(t_n) = e^{-it_n^1\Delta^2}\psi^1 + W_n^M. \tag{5.17}$$

Also as in the proof of Proposition 5.5, we obtain that $M(\psi^1) = 1$, $\lim_{n\to+\infty} E(e^{-it_n^1\Delta^2}\psi^1) = (M^{\frac{2-s_c}{s_c}}E)_c$, $\lim_{n\to+\infty} M(W_n^M) = 0$ and $\lim_{n\to+\infty} E(W_n^M) = 0$. Thus by Lemma 4.3, we get

$$\lim_{n\to+\infty} \|W_n^M\|_{H^2} = 0. \tag{5.18}$$

We claim now that t_n^1 converges to some finite t^1 up to a subsequence, and then, since $e^{-it_n^1\Delta^2}\psi^1 \to e^{-it^1\Delta^2}\psi^1$ in H^2, (5.18) implies that $u(t_n)$ converges in H^2, concluding our proof. It suffices to show the above claim. Contrarily, if $t_n^1 \to -\infty$, then

$$\|e^{it\Delta^2}u(t_n)\|_{Z([0,\infty))} \leq \|e^{i(t-t_n^1)\Delta^2}\psi^1\|_{Z([0,\infty))} + \|e^{it\Delta^2}W_n^M\|_{Z([0,\infty))}.$$

Note that

$$\lim_{n\to+\infty} \|e^{i(t-t_n^1)\Delta^2}\psi^1\|_{Z([0,\infty))} = \lim_{n\to+\infty} \|e^{it\Delta^2}\psi^1\|_{Z([-t_n^1,\infty))} = 0$$

and $\|e^{it\Delta^2}W_n^M\|_{Z([0,\infty))} \leq \frac{1}{2}\delta_{sd}$ by taking n sufficiently large, in contradiction to the small data scattering theory. If, on the other hand, $t_n^1 \to +\infty$, we will similarly have $\|e^{it\Delta^2}u(t_n)\|_{Z((-\infty,0])} \leq \frac{1}{2}\delta_{sd}$. Thus the small data scattering theory (Proposition 2.2) shows that $\|u\|_{Z((-\infty,t_n])} \leq \delta_{sd}$. Since $t_n \to +\infty$, by sending $n \to +\infty$, we obtain $\|u\|_{Z((-\infty,+\infty))} \leq \delta_{sd}$, which is a contradiction again.

□

Corollary 5.7. *Let u be a solution of (1.2) such that $K = \{u(t)|\ t \in [0, +\infty)\}$ is precompact in H^2. Then for each $\epsilon > 0$, there exists $R > 0$ independent of t such that*

$$\int_{|x|>R} |\Delta u(x,t)|^2 + |u(x,t)|^2 + |u(x,t)|^{p+1} dx \le \epsilon.$$

Proof. Contrarily, if not, then there exists $\epsilon_0 > 0$ and a sequence $t_n \to +\infty$ such that, for each n,

$$\int_{|x|>n} |\Delta u(x,t_n)|^2 + |u(x,t_n)|^2 + |u(x,t_n)|^{p+1} dx \ge 2\epsilon_0.$$

By the precompactness of K, there exists some $\phi \in H^2$ such that, up to a subsequence of t_n, $u(t_n) \to \phi$ in H^2. Thus taking n large, we obtain

$$\int_{|x|>n} |\Delta \phi(x)|^2 + |\phi(x)|^2 + |\phi(x)|^{p+1} dx \ge \epsilon_0. \tag{5.19}$$

On the other hand, since $\phi \in H^2$ and $\|\phi\|_{p+1} \le c\|\phi\|_{H^2}$ by Sobolev embedding, by taking n sufficiently large, we have

$$\int_{|x|>n} |\Delta \phi(x)|^2 + |\phi(x)|^2 + |\phi(x)|^{p+1} dx \le \frac{1}{2}\epsilon_0,$$

in contradiction to (5.19).

□

6. A Rigidity theorem

In this section, we prove the following statement and finish the proof of Theorem 1.1.

Theorem 6.1. *Assume $u_0 \in H^2$ is radial satisfying $\|u_0\|_2 = 1$, $E(u_0) < E(Q)M(Q)^{\frac{2-s_c}{s_c}}$ and $\|\Delta u_0\|_2 < \|Q\|_2^{\frac{2-s_c}{s_c}} \|\Delta Q\|_2$. Let u be the corresponding solution of (1.2) with initial data u_0. If $K_+ = \{u(t) : t \in [0,\infty)\}$ is precompact in H^2, then $u_0 \equiv 0$. The same conclusion holds if $K_- = \{u(t) : t \in (-\infty, 0]\}$ is precompact in H^2.*

Remark 6.2. In view of Proposition 5.6, Theorem 6.1 implies that u_c obtained in Proposition 5.5 cannot exist. Thus, there must holds that $(M^{\frac{2-s_c}{s_c}} E)_c = E(Q)M(Q)^{\frac{2-s_c}{s_c}}$, which combined with Proposition 2.3 implies Theorem 1.1. Then, it suffices to show Theorem 6.1.

Proof. Define

$$A_R(t) \equiv Im \int x\phi(\frac{|x|}{R}) \cdot \nabla u \bar{u} dx, \tag{6.1}$$

where $\phi(r) \in C_c^\infty(\mathbb{R})$ is equal to 1 when $r \le 1$, and to 0 when $r \ge 2$. Then for the radial solution u of the problem (1.2), we compute that

$$A'_R(t) = Im \int \nabla \left(x\phi(\frac{x}{R}) \right) u_t \bar{u} dx + 2Im \int x\phi(\frac{x}{R}) \cdot \nabla u \bar{u}_t \equiv I_1 + I_2,$$

where

$$
\begin{aligned}
I_1 =& - Re\int \nabla\left(x\phi(\frac{x}{R})\right)\bar{u}(\Delta^2 u - |u|^{p-1}u)dx\\
=& - N\int |\Delta u|^2 - |u|^{p+1}dx + N\int \left(1-\phi(\frac{x}{R})\right)\left(|\Delta u|^2 - |u|^{p+1}\right)dx\\
& - Re\int \frac{x}{R}\cdot\nabla(\frac{x}{R})\left(\bar{u}\Delta^2 u - |u|^{p+1}\right)dx\\
=& - N\int |\Delta u|^2 - |u|^{p+1}dx + Res_1
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 =& - 2Re\int x\phi(\frac{x}{R})\cdot\nabla\bar{u}(\Delta^2 u - |u|^{p-1}u)dx\\
=& - 2Re\int x\phi(\frac{x}{R})\cdot\nabla\bar{u}\Delta^2 u dx - \frac{2}{p+1}\int\nabla\left(x\phi(\frac{x}{R})\right)|u|^{p+1}dx\\
=&Re\int\nabla\left(x\phi(\frac{x}{R})\right)|\Delta u|^2 dx - 2Re\int\partial_k^2(\phi x_j)\partial_j\bar{u}\Delta u dx\\
& - 4Re\int\partial_k(\phi x_j)\partial_j\partial_k\bar{u}\Delta u dx - \frac{2}{p+1}\int\nabla\left(x\phi(\frac{x}{R})\right)|u|^{p+1}dx\\
=&N\int|\Delta u|^2 dx + N\int(\phi(\frac{x}{R})-1)|\Delta u|^2 dx + Re\int\frac{x}{R}\cdot\nabla\phi(\frac{x}{R})|\Delta u|^2 dx\\
& - 2Re\int\partial_k^2(\phi x_j)\partial_j\bar{u}\Delta u dx - 4\int|\Delta u|^2 dx + 4\int(\phi(\frac{x}{R})-1)|\Delta u|^2 dx\\
& - 4Re\int\partial_k\phi(\frac{x}{R})\partial x_j R\partial_j\partial_k\bar{u}\Delta u - \frac{2}{p+1}\int\nabla\left(x\phi(\frac{x}{R})\right)|u|^{p+1}dx\\
=&(N-4)\int|\Delta u|^2 dx - \frac{2N}{p+1}\int|u|^{p+1}dx\\
& + \frac{2N}{p+1}\int(1-\phi)|u|^{p+1}dx - \frac{2}{p+1}\int\frac{x}{R}\cdot\nabla\phi|u|^{p+1}dx\\
=&(N-4)\int|\Delta u|^2 dx - \frac{2N}{p+1}\int|u|^{p+1}dx + Res_2.
\end{aligned}
$$

From Corollary 5.7, we can infer that $|Res| = |Res_1 + Res_2| = o_R(1) \to 0$ as $R \to \infty$, uniformly in $t \in [0,\infty)$. Finally, we have

$$
A_R'(t) = -4\int|\Delta u|^2 dx + \frac{N(p-1)}{p+1}\int|u|^{p+1}dx + o_R(1), \tag{6.2}
$$

where $o_R(1) \to 0$ as $R \to +\infty$ uniformly in t.

Let a positive constant $\delta \in (0,1)$ be such that $E(u_0) < (1-\delta)E(Q)M(Q)^{\frac{2-s_c}{s_c}}$. By Lemma 4.2 and Lemma 4.3, we obtain that there exists some constant $\delta_0 > 0$ such that

$$-4\int |\Delta u|^2 dx + \frac{N(p-1)}{p+1}\int |u|^{p+1} dx \leq -2\delta_0 \int |\Delta u_0|^2 dx,$$

which implies by (6.2) that $A_R'(t) \leq -\delta_0 \int |\Delta u_0|^2 dx$ for R sufficiently large. Thus, we have

$$A_R(t) - A_R(0) \leq -\delta_0 t \int |\Delta u_0|^2 dx.$$

On the other hand, by the definition of $A_R(t)$, we should have

$$|A_R(t) - A_R(0)| \leq C_R \|Q\|_{H^2}^2,$$

which is a contradiction for t large unless $u_0 = 0$.

□

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